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TEACHING INCOMMENSURABLES

The teaching of incommensurables is largely a matter of expediency. If a class recognizes that similar quantities may or may not have a common measure, and if it questions whether theorems true in the first case are also true in the second, then that class is in a position to follow the proof of the incommensurable cases with appreciation, even though no one of the class can reproduce it independently. The fact that the proof exists and that the pupils have followed it is the important thing.

If, on the other hand, the class is too immature to realize that incommensurables do exist, the proof of such cases is best postponed until the geometry is reviewed. A theorem that seems useless on the first reading of the subject, often slips into place on the second.

Unless Legendre's proof be used, the notion of incommensurables implies that of limits. To my mind this concept can well be introduced here. It develops the idea of variables implied in the work on formulæ and the graphic treatment of equations; it leads to an extended treatment in the Calculus. The idea of a variable that tends toward a limit is familiar—the ripple when a stone is dropped into a pond, the creation of a partial vacuum, the disintegration of radium, the height of a growing boy. All that really is needed is to clarify our notions of variables by defining the conditions under which they may be said to approach limits.

Geometric illustrations of this definition have the great advantage of resting on accepted postulates; for many of the so-called practical illustrations require more time than the illustration justifies for the preliminary discussion that sets up the problem.

In a search for a quantitative geometric illustration of limits I came upon the following. The figure is used in another connection by Maxwell in his Non-Euclidean Geometry p. 27.

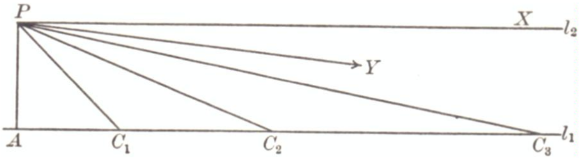

FIG. 1.

Suppose l_1 and l_2 are each perpendicular to PA.

$$\begin{aligned}\text{Let } AC_1 &= PA,\\ C_1C_2 &= PC_1,\\ C_2C_3 &= PC_2, \text{ etc.}\end{aligned}$$

There is no limit to the number of points C_1, C_2, C_3, for a line may be extended for any required length—in particular to the length PC_i.

This gives an infinite series of points on l_1 and also an infinite series of lines PC_i each greater than the preceding one. The corresponding segments AC_1, AC_2 therefore increase without limit.

Here, then, is one of our illustrations.

Corresponding to each point C_i there is an angle PC_iA. An infinite series of these points gives rise to an infinite series of these angles and each angle is one half of its predecessor

Each of these angles is complementary to the angle APC_i.

If the angle PC_iA becomes equal to zero, the angle $APC_i =$ 90° and PC_i is parallel to AC_i, which is impossible. Therefore the angle PC_iA cannot become equal to zero, and by the principle of continuity it cannot become less than zero. It is evident that however small an angle YPX we may draw, some line PC_i of the series can be found longer than PY. Then the angle PC_iA will be less than the angle PYA. Thus the angle PCA approaches zero as its limit, and similarly the angle APC approaches 90° as its limit.

The angle PCA equals the angle CPX.

Now if the limits of these two angles are not equal, one of two things must be true. Either l_1 has ceased to be parallel to l_2 which is contrary to our hypothesis; or alternate interior angles of parallel lines have ceased to be equal, which is inconceivable. This figure, then, illustrates a variable which becomes infinite, a variable that approaches zero, and two constantly equal variables approaching their limits.

This detail cannot, of course, be given to a class in toto, but the possibilities of the figure appeal to a class and much of the work can be developed by the pupils in a short period. It also helps in the realization of the essential normalness of our definition of a limit.

Legendre's proof of incommensurable cases may be interesting in this connection. It is an exquisite example of a reductio ad absurdum. He applies this proof to the rectangles having equal altitudes and to the central angle proposition. He proves the proportional division of the sides of a triangle by a different method, although the method of the other cases may readily be extended to this one.

Let us take the proposition of the rectangles having equal altitudes (Legendre Book III., prop. 3).

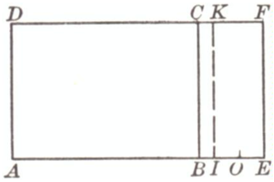

FIG. 2.

If the bases AB and AE of the rectangles $ABCD$, $AEFD$ were commensurable we would know

$$\frac{ABCD}{AEFD}=\frac{AB}{AE}$$

If AB and AE have no common measure, it may be necessary to substitute for AB a quantity AO to have a true proportion.

Then (1) $\frac{ABCD}{AEFD}=\frac{AO}{AE}$

AO may be greater or less than AB.

Suppose AO is greater than AB. Then O lies on AE between B and E.

Take some unit less than BO which will be contained a whole number of times in AE. Mark this unit along AE. At least one point of division, I, will come between B and O. At I draw $IK \perp AE$.

AI and AE are commensurable by construction.

$$\therefore (2) \quad \frac{AIKD}{AEFD} = \frac{AI}{AE}.$$

Divide (1) by (2) $\dfrac{ABCD}{AIKD} = \dfrac{AO}{AI}$,

but the first of these ratios is less than one, the second greater than one, therefore the proportion is untrue and the hypothesis that AO is greater than AB is absurd. Similarly AO cannot be less than AB.

$\therefore AO = AB$ and, substituting in (1),

$$\frac{ABCD}{AEFD} = \frac{AB}{AE}.$$

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